PROBABILITY AND STATISTICS FOR ENGINEERS MIDTERM 1		
Code : *CVE 303* Acad.Year : *2019-20* Semester : *Fall* Date : *03.11.2019* Time : *10:40* Duration : *110 min*	Last Name: Name: Solutions Student ID: Signature: 7 QUESTIONS ON 4 PAGES TOTAL 100 POINTS	#:

P1. (25)	P2. (25)	P3. (25)	P4. (25)		Total. (100)

1. Use the following data to fill in the table of probabilities and answer the questions below.

- A test for an event has a **false positive** rate of $\frac{1}{10}$ $P(\text{Test Y} \mid \text{Event N}) = 1/10$
- The test has a **false negative** rate of $\frac{1}{20}$ $P(\text{Test N} \mid \text{Event Y}) = 1/20$
- Prior to testing, the probability of the event occurring is estimated to be $\frac{4}{10}$ $P(\text{Event Y}) = 4/10$

$P(\text{Test N} \mid \text{Event Y}) = 1/20$ $P(\text{Event Y}) = 4/10$

Prob.	Test Y	Test N	Total
Event Y	38/100	2/100	4/10
Event N	6/100	54/100	6/10
Total	44/100	56/100	1

$P(\text{Test Y} \mid \text{Event N}) = 1/10$

- What is the probability the event occurring **and** the test result being positive?

$P(\text{Event Y } \underline{\text{and}} \text{ Test Y}) =$

$\boxed{38/100}$

- What is the probability the event occurring **or** the test result being positive?

$P(\text{Event Y } \underline{\text{or}} \text{ Test Y}) =$

$1 - 54/100 = \boxed{46/100}$

- What is the probability of the test being positive **if** the event occurred?

$P(\text{Test Y} \mid \text{Event Y}) =$

$\dfrac{38/100}{4/10} = \dfrac{38}{40} = \boxed{\dfrac{19}{20}}$

Note: This is $1 - P(\text{Test N} \mid \text{Event Y}) = 1 - 1/20$

- What is the probability of a positive result from the test?

$P(\text{Test Y}) =$

$\boxed{44/100}$

- What is the probability of the event having occurred **if** the test result was positive?

$P(\text{Event Y} \mid \text{Test Y}) =$

$\dfrac{38/100}{44/100} = \dfrac{38}{44} = \boxed{\dfrac{19}{22}}$

- What is the probability of the event having occurred **if** the test result was negative?

$P(\text{Event Y} \mid \text{Test N}) =$

$\dfrac{2/100}{56/100} = \dfrac{2}{56} = \boxed{\dfrac{1}{28}}$

2. Consider the experiment where you flip a fair coin two times. Let X be the number of times you get "Heads" from the two flips.

Result		X
$\{TT\}$	$\longrightarrow$	$X=0$
$\{TH, HT\}$	$\longrightarrow$	$X=1$
$\{HH\}$	$\longrightarrow$	$X=2$

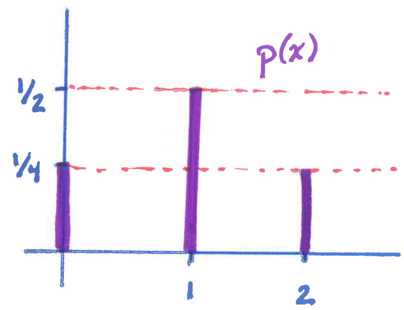
- Give all possible outcomes for X.

$$\boxed{X = \{0, 1, 2\}}$$

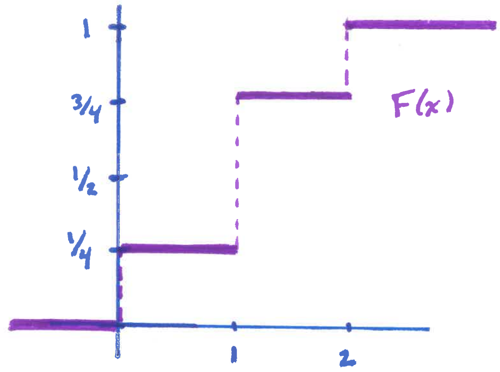
- Make a table with the pmf of X.

X	0	1	2
$p(x)$	$1/4$	$1/2$	$1/4$

- Draw a rough graph of the **pmf** of X. Label important values on the x and y axis.



- Draw a rough graph of the **cdf** of X. Label important values on the x and y axis.



3. The discrete random variable X has the probability mass function given to the right.

x	-1	0	1	2
$p(x)$	$\frac{4}{10}$	$\frac{2}{10}$	$\frac{1}{10}$	$\frac{3}{10}$

Give the following values. (Do not reduce fractions.)

- $P(X > 0) = 1/10 + 3/10 = \boxed{4/10}$
- $P(-1 < X < 2) = 2/10 + 1/10 = \boxed{3/10}$
- $E[X] = (-1)\,4/10 + (0)\,2/10 + (1)\,1/10 + (2)\,3/10 = \frac{-4+1+6}{10} = \boxed{3/10}$
- $E[X^2] = (-1)^2\,4/10 + (0)^2\,2/10 + (1)^2\,1/10 + (2)^2\,3/10 = \frac{4+1+12}{10} = \boxed{17/10}$
- $\mathrm{Var}[X] = E[X^2] - (E[X])^2 = 17/10 - (3/10)^2 = \frac{170-9}{100} = \boxed{161/100}$
- $E[2X+3] = 2E[X] + 3 = 2 \cdot 3/10 + 3 = \boxed{36/10}$
- $\mathrm{Var}[2X+3] = 2^2\,\mathrm{Var}[X] = 4 \cdot 161/100 = \boxed{644/100}$

4. Convert between $X \sim \text{Normal}(\mu = 3,\ \sigma = 2)$ and standard normal Z.

- $P(-1 \leqslant X \leqslant 2) = P\left(\boxed{\frac{-1-3}{2} = -2} \leqslant Z \leqslant \boxed{\frac{2-3}{2} = -\frac{1}{2}}\right)$

- $P(X \leqslant x) = P\left(Z \leqslant \boxed{\frac{x-3}{2}}\right)$

- $P(1 \leqslant Z \leqslant 2) = P\left(\boxed{1 \cdot 2 + 3 = 5} \leqslant X \leqslant \boxed{2 \cdot 2 + 3 = 7}\right)$

- $P(Z \leqslant z) = P\left(X \leqslant \boxed{2z + 3}\right)$

5. In the parts below, assume that $X \sim \text{Normal}(\mu = 3,\ \sigma = 2)$.

Express the following probabilities using "pnorm(x, μ, σ)" $=$ $P(X < x)$, or "pnorm(z)".

- $P(X > 2) = 1 - P(X \leq 2)$

  $\boxed{1 - \text{pnorm}(2, 3, 2)}$

  $= 1 - \text{pnorm}\left(\frac{2-3}{2}\right)$

- $P(|X - 3| > 2) = 2 \cdot P(X - 3 < -2)$

  $2 \cdot \text{pnorm}(1, 3, 2)$

  $= \boxed{2 \cdot \text{pnorm}(-2, 0, 2)}$

  $= 2 \cdot \text{pnorm}(-\frac{2}{2})$

Express the following critical values using "qnorm(a, μ, σ)" $= x_a$ where $P(X < x_a) = a$.

Example: If $P(X < x) = .2$ then $x =$ qnorm(.2, 3, 2).

- If $P(X > x) = .05$

  then $x =$ $\boxed{1 - \text{pnorm}(x, 3, 2) = .05}$

  $\text{pnorm}(x, 3, 2) = .95$

  $\boxed{x = \text{qnorm}(.95, 3, 2)}$

  $= \text{qnorm}(.95) \cdot 2 + 3$

- If $P(|X - 3| > x) = .05$

  then $x =$ $\boxed{2 \cdot \text{pnorm}(-x, 0, 2) = .05}$

  $\text{pnorm}(-x, 0, 2) = .025$

  $\boxed{x = -\text{qnorm}(.025, 0, 2)}$

  $= -\text{qnorm}(.025) \cdot 2$

  $= 3 - \text{qnorm}(.025, 3, 2)$

Give short answers to the questions below.

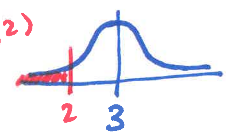
- Is pnorm(2, 3, 2) **bigger** or **smaller** than ½? Why?

  pnorm(2, 3, 2) < ½

  because pnorm(x, 3, 2) = P(X ≤ x)

  where mean of X is 3

  area is pnorm(2,3,2)

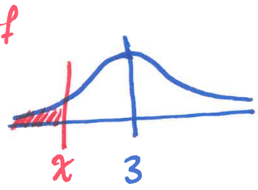


- Is qnorm(.2, 3, 2) **bigger** or **smaller** than 3? Why?

  qnorm(.2, 3, 2) < 3

  because qnorm(.5, 3, 2) = 3 since 3 is mean of X

  x<3 if area<½



6. The discrete joint random variable (X, Y) has joint pmf given to the right.

Compute the following.

$p(x,y)$	$x=0$	$x=1$	$x=2$	$p_Y(y)$
$y=0$	$\frac{3}{15}$	$\frac{1}{15}$	$\frac{1}{15}$	$5/15$
$y=1$	$\frac{2}{15}$	$\frac{1}{15}$	$\frac{3}{15}$	$6/15$
$y=2$	$\frac{2}{15}$	$\frac{1}{15}$	$\frac{1}{15}$	$4/15$
$p_X(x)$	$7/15$	$3/15$	$5/15$	

$p(y\,|\,2)$

$p(x\,|\,1)$

- The marginal pmf for X and Y.

x	0	1	2
$p_X(x)$	$\boxed{7/15}$	$\boxed{3/15}$	$\boxed{5/15}$

y	0	1	2
$p_Y(y)$	$\boxed{5/15}$	$\boxed{6/15}$	$\boxed{4/15}$

- The conditional pmf for $(X\,|\,Y=1)$ and $(Y\,|\,X=2)$.

x	0	1	2
$p_{X\|Y}(x\,\|\,1)$	$\boxed{2/6}$	$\boxed{1/6}$	$\boxed{3/6}$
	$\frac{2/15}{6/15}$	$\frac{1/15}{6/15}$	$\frac{3/15}{6/15}$

y	0	1	2
$p_{Y\|X}(y\,\|\,2)$	$\boxed{1/5}$	$\boxed{3/5}$	$\boxed{1/5}$
	$\frac{1/15}{5/15}$	$\frac{3/15}{5/15}$	$\frac{1/15}{5/15}$

Compute expected values of the functions below. (Do not simplify fractions.)

- $\mathrm{E}[X] =$

$(0)\,7/15 + (1)\,3/15 + (2)\,5/15$

$= \boxed{13/15}$

- $\mathrm{E}[XY] =$

$$\left.\begin{aligned} &\cancel{(0\cdot0)\,3/15} + \cancel{(0\cdot1)\,1/15} + \cancel{(0\cdot2)\,1/15} \\ &+ \cancel{(1\cdot0)\,2/15} + (1\cdot1)\,1/15 + (1\cdot2)\,3/15 \\ &+ \cancel{(2\cdot0)\,2/15} + (2\cdot1)\,1/15 + (2\cdot2)\,1/15 \end{aligned}\right\} = \boxed{13/15}$$

7. Give short answers below. Suppose X, Y is a joint random variable with pmf $p(x,y)$. These questions are unrelated to the problem above.

- What probability is computed by the **joint pmf** $p(1,2)$?

$p(1,2) = P(X=1 \text{ and } Y=2)$

"probability that $X=1$ and $Y=2$"

- What probability is computed by the **marginal pmf** $p_X(1)$?

$p_X(1) = P(X=1)$

"probability that $X=1$"

(no restrictions on Y)

- What probability is computed by the **conditional pmf** $p_{X|Y}(1\,|\,2)$?

$p_{X|Y}(1\,|\,2) = P(X=1 \mid Y=2)$

$$= \frac{P(X=1 \text{ and } Y=2)}{P(Y=2)}$$

"probability that $X=1$ if $Y=2$"

or "$X=1$ given that $Y=2$"

- How can you tell if X and Y are independent using pmf?

X and Y are independent if

$p(x,y) = p_X(x)\cdot p_Y(y)$

"$P(X=x \text{ and } Y=y) = P(X=x)\cdot P(Y=y)$"

for all x and y.